Maximilien Péroux.
17 October 2018.
Homotopy theory for coalgebras.

I. Defs.

$\mathcal{C}$ symmetric monoidal cat,

$R \in \mathrm{CAlg}(\mathcal{C})$.

$(\mathrm{Mod}_R(\mathcal{C}), \otimes_R, R)$

$\mathrm{coAlg}_R(\mathcal{C}) := \mathrm{coMon}(\mathrm{Mod}_R(\mathcal{C}))$.

C.

$\Delta : C \to C \otimes_R C$ comultiplication.

$C \to R$ counit.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "1 \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes 1"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, equal] & C \otimes C \arrow[d, "1 \otimes \varepsilon"] \\ C \otimes C \arrow[r, "\varepsilon \otimes 1"'] & C \end{tikzcd}$$

C is <u>cocommutative</u> if

$$\begin{tikzcd} C \otimes C \arrow[rr, "\text{switch}"] & & C \otimes C \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

<u>Rem</u>. $\mathrm{CoAlg}_R(\mathcal{C}) \simeq \mathrm{Alg}_R(\mathrm{Mod}_R^{op})^{op}$.

<u>Alg</u>	<u>coAlg</u>.
$+, \times$	factorization,
concatenation	partition,
composition	decomposition

<u>Ex</u>. $k[x]$

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k$$

$\varepsilon : k[x] \to k$.

Def. $\mathcal{O}$ an ∞-operad, $\mathcal{C}$ an $\mathcal{O}$-monoidal ∞-cat.

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[d, dash, "\text{coCartesian}"] & & \\ \mathcal{O}^{\otimes} \arrow[r] & \widehat{\mathrm{Cat}}_\infty \arrow[r, phantom, "\underset{op}{\simeq}"] & \widehat{\mathrm{Cat}}_\infty \end{tikzcd}$$

Unstraighten.
Get an $\mathcal{O}$-monoidal structure on $\mathcal{C}^{op}$.

Ex. If $\mathcal{C}^{\times}$ is cartesian, every object has a unique coalgebra structure $X \xrightarrow{\Delta} X \times X$, $X \to *$.

Ex. codga $=$ coMon(ch_R).

X a space, k a field.

X a space, k a field,

$$C_*(X,k) \xrightarrow{\Delta} C_*(X \times X, k) \xleftarrow{\simeq} C_*(X,k) \otimes C_*(X,k)$$

E-Z

(strict)
is a $\sim$ coalgebra.

Ex. $H\mathbb{F}_p * H\mathbb{F}_p$, $MU_* MU$ } Hopf algebras.

... Hopf Galois extensions.

$\mathcal{C} = \mathrm{Set}, s\mathrm{Set}, \mathrm{Top}$

<u>Prop</u> (P. Shipley). $(-)_+ : \mathcal{C} \to \mathcal{C}_*$

$X \mapsto X_+$

lifts to an $\simeq$ of cat.

$$\mathrm{coAlg}(\mathcal{C}) \simeq \mathrm{coAlg}(\mathcal{C}_*).$$

II. Sadness.

$\mathcal{C}$ loc. presentable $\Rightarrow$ $\mathcal{C}^{op}$ not typ. presentable.

<u>But</u>, $\mathrm{coAlg}(\mathcal{C})$ is presentable if

$\mathcal{C}$ is presentably sym. mon.

$$\mathcal{C} \underset{T}{\overset{U}{\rightleftarrows}} \mathrm{Alg}(\mathcal{C})$$

$$T(X) \simeq \coprod_{n \geq 0} X^{\otimes n}$$

$$\mathrm{coAlg}(\mathcal{C}) \underset{}{\overset{T^\vee}{\underset{\text{cofree}}{\rightleftarrows}}} \mathcal{C}$$

$T^\vee(X) = ?$

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{class. a}", "\text{coAlg str.}"'] \arrow[dr, equal] & & T^\vee(X) \arrow[dl, "\text{counit}"] \\ & X & \end{tikzcd}$$

So, $T^\vee$ has to have <u>all</u> counits.

Also, $\otimes$ bad. Limits of of coAlg are hard.

Model cat story not so good.
- inj. model str.
- inj. res.
- fibrantly gen is my con.
- not a monoidal model cat.

$$\begin{tikzcd} T^\vee(X) \subseteq \prod_{n \geq 1} X^{\otimes n} \arrow[r] \arrow[d, "(*)"] & \prod_{\substack{a+b=n,\\ a,b \geq 1}} X^{\otimes a} \otimes X^{\otimes b} \\ \prod X^{\otimes a} \otimes \prod X^{\otimes b} \arrow[ur, "\text{exists}"'] & \end{tikzcd}$$

Largest sst. + exists.

Doesn't typ. exist.

$\mathcal{C} = \mathrm{Ab}, \mathrm{Ch}, \mathrm{Sp}$

III. coAlg in Sp.

Prop (P.) Sp^Σ = symmetric spectra.

$$\mathrm{coAlg}(\mathrm{Mod}_R(Sp^\Sigma)).$$

W = stable eqs.

cof = monos.

Rm But, $\mathrm{Ho}(\mathrm{coAlg}_R(Sp^\Sigma))$ is not what we want.

Thm (P.–Shipley). In Sp^Σ, Sp^O, EKMM, ...,
any $\mathbb{S}$ coalgebra is cocommutativity.

More gen, if R is a commutative $\mathbb{S}$-algebra
with $R_0 \cong S^0$, then any R-coAlgebra is cocommutative and induced.

Not what we want?

$$\mathrm{Ho}(\mathrm{coAlg}_{A_\infty}(Sp^\infty))$$

A finite coalgebra, noncommutative.

DA a $\mathbb{S}$-coalgebra noncommutative.

IV. Rectification.

$$N(\mathrm{Alg}(\mathcal{C}))[W^{-1}] \simeq \mathrm{Alg}_{A_\infty}(N(\mathcal{C})[W^{-1}]).$$

$$\mathcal{C} = \mathrm{Ch}, \mathrm{Sp}.$$

Issue

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr, dashed, "?"'] & \mathcal{C} \otimes \mathcal{C} \\ & \mathcal{C} \otimes^{L} \mathcal{C} \arrow[u] \end{tikzcd}$$

might not exist

$\mathcal{C} = \mathrm{Ch}_k$, k = field.

$$N(\mathrm{coAlg}(\mathrm{Ch}_k))[W^{-1}] \underset{???}{\simeq} \mathrm{coAlg}_{A_\infty}(N(\mathrm{Ch}_k)[W^{-1}]).$$